数理解析研究所講究録 1409 巻 2005 年 115-124

Optimal Planning For Failure-Prone Manufacturing Systems: A Discrete Nonlinear Optimization Approach

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Abstract
The paper addresses a generalized discrete time EMQ (Economic Manufacturing Quantity) model for a failure-prone manufacturing system consisting of a single machine producing a single part type. The time to failure of the machine, corrective and preventive repair times all are assumed to follow arbitrary discrete probability distributions. The proposed model is formulated based on the net present value (NPV) approach, treating the machine production rate as (i) predetermined or, (ii) flexible. The criteria for the existence of a local optimal solution for the NPV model and the long-run average cost model are derived under geometric repair time distributions, when the production rate is predetermined. A hybridized neural network (NN) and branch and bound (BB) algorithm is developed for the associated discrete nonlinear optimization problem when the production rate is treated as flexible. Numerical examples are provided to determine the optimal production policy numerically and examine its dependence on the stochastic behavior of the system failure and repair. A comparison of the outcome obtained by time discretization in the corresponding continuous time model to that of the discrete time model is also made.

1. Introduction

Most of the production planning models typically attempt to derive the optimal production policy by minimizing the system costs disregarding the impact of machine failure. Yet, an unexpected machine failure can result loss of revenue due to down time, missed delivery schedules, poor product quality, lower product yield, cost of repairing and so on. The issue of interdependence between production and maintenance for a stochastically failing equipment was raised first by McCall (1965). Bielecki and Kumar (1988) showed that there exists a range of parameter values describing an unreliable manufacturing system for which zero inventory policy is exactly optimal even when the production capacity is uncertain. Groenevelt *et al.* (1992a) analyzed the impacts of machine breakdown and corrective maintenance on an economic manufacturing quantity (EMQ) model, assuming exponentially distributed inter-failure time and instantaneous repair time. They showed that the optimal production lot size is greater than that of the classical EMQ model. In the subsequent article (1992b), they investigated the issue of safety stocks required to meet a managerially prescribed service level under a simplified assumption of exponential failure time and randomly distributed repair time. Kim and Hong (1997) generalized the results of Groenevelt *et al.* (1992a) assuming arbitrarily distributed inter-failure time. For general failure and general repair time distributions, Dohi *et al.* (1997) determined the optimal production policy which can be characterized as an age replenishment like policy. Makis and Fung (1998) studied the joint effect of process deterioration and machine breakdowns on the optimal lot size and the optimal number of inspections in a production cycle. Liu and Cao (1999) developed an unreliable EMQ model where the demand is assumed to be a compound Poisson process. Cheung and Hausmann (1997), Dohi *et al.* (2001) investigated the joint implementation of preventive maintenance and safety stocks in an unreliable production environment.

The EMQ models with stochastic machine breakdown and repair have been developed so far based on continuous failure time distributions. However, the time to failure of a unit might be discrete in many practical situations. For example, consider the failure of electronic circuits, switching devices, electromagnetic devices etc. where the time to failure, would be better measured by the number of cycles to failure rather than the instance of occurrence. So, it is appropriate to deal such failure events with discrete time failure distributions. Abboud (2001) modeled an unreliable single machine production-inventory system as a discrete-time Markov chain, assuming geometric failure and geometric repair time distributions.

This article focuses on an EMQ problem for an unreliable manufacturing system in discrete time setting under a framework in which the time to machine failure, corrective and preventive times are assumed to follow arbitrary discrete probability distributions. In the following section, the NPV model with general failure and general repair time distributions in discrete time setting is presented. The criteria for the existence of a local optimal solution are derived under geometric repair time distributions when the production rate is predetermined. In Section 3, the traditional long-run average cost model is derived from the NPV model by taking limitation on discount rate. Section 4 deals with the associated discrete nonlinear integer programing (NIP) problem and development of its solution algorithm. Section 5 is devoted to numerical illustrations of the lot sizing policy, sensitivity analysis and comparison of the outcome of the discrete time model and that obtained by time discretization in the corresponding continuous time model. Finally, the concluding remarks and future research directions are presented in Section 6.

2. The NPV Model

Notations:

n = discrete time point, $n = 0, 1, 2, \cdots\cdots$

$P(n)$ = discrete failure time distribution with p.m.f. $p(n)$

$\overline{\psi}(\cdot)$ = survivor function of the function $\psi(\cdot)$, *i.e.*, $\overline{\psi}(\cdot) = 1 - \psi(\cdot)$

$G_1(l_1)$ = discrete corrective repair time distribution with p.m.f. $g_1(l_1)$ and finite mean $1/\mu_1$ (> 0)

$G_2(l_2)$ = discrete preventive repair time distribution with p.m.f. $g_2(l_2)$ and finite mean $1/\mu_2$ (> 0)

d (> 0) = demand rate

p $(> d)$ = production rate

c_0 (> 0) = setup cost

c_1 (> 0) = corrective repair cost per unit time

c_2 (> 0) = preventive repair cost per unit time
c_i (> 0) = inventory holding cost per unit product per unit time
c_s (> 0) = shortage penalty cost per unit product
b $(0 < b < 1)$ = discount factor (rate)

Model formulation when the production rate is predetermined:

Consider a single-unit single-item production system which starts at time $n = 0$. If the machine does not fail up to a prescribed production time $n_0 \in (0, \infty)$ then the production is stopped and preventive repair is carried out to return back the machine to the same initial working condition before the start of the next production cycle. If, however, the machine fails before time n_0, then the corrective repair is started immediately. During corrective/preventive repair, the demands are met first from the accumulated inventory. If the on-hand stock is sufficient to meet up demands during machine repair then the production process is started only when the on-hand stock is exhausted. Since the corrective and preventive repair times are assumed to be random, there is a possibility that the on-hand stock may exhaust before repair is completed. In such a case, the unsatisfied demands are not met after machine repair and are assumed to be lost.

During production phase, the increment of on-hand stock level after one discrete time unit is $p-d$. For discrete time setting, we assume the production rate p such that $p = kd$, where k is a known integer greater than 1. This production demand relationship ensures that the on-hand stock after a production run or a machine failure will be completely exhausted in some future time units. Under this assumption, the NPV of the expected inventory holding cost per cycle is given by

$$\begin{aligned} H_c(n_0) &= c_i d\Big[\sum_{n=0}^{n_0-1} \Big\{ \sum_{i=0}^{n-1} (k-1)i\, b^i + \sum_{i=n}^{kn} (kn-i) b^i \Big\} p(n) + \sum_{n=n_0}^{\infty} \Big\{ \sum_{j=0}^{n_0-1} (k-1)j\, b^j \\ &\quad + \sum_{j=n_0}^{kn_0} (kn_0 - j) b^j \Big\} p(n) \Big] \end{aligned}$$

Similarly, the NPV of the expected shortage cost per cycle is

$$\begin{aligned} S_c(n_0) &= c_s d \Big[\sum_{n=0}^{n_0-1} \sum_{l_1=(k-1)n+1}^{\infty} \sum_{i=0}^{l_1-(k-1)n-1} b^{kn+i} g_1(l_1)\, p(n) \\ &\quad + \sum_{n=n_0}^{\infty} \sum_{l_2=(k-1)n_0+1}^{\infty} \sum_{j=0}^{l_2-(k-1)n_0-1} b^{kn_0+j} g_2(l_2)\, p(n) \Big] \end{aligned}$$

and the NPV of the expected repair costs for one cycle is

$$R_c(n_0) = c_1 \sum_{n=0}^{n_0-1} \sum_{l_1=0}^{\infty} \sum_{i=0}^{l_1-1} b^{n+i} g_1(l_1)\, p(n) + c_2 \sum_{n=n_0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{j=0}^{l_2-1} b^{n_0+j} g_2(l_2)\, p(n)$$

Hence the NPV of the expected total cost for one cycle is

$$S_b(n_0) = c_0 + H_c(n_0) + S_c(n_0) + R_c(n_0). \tag{1}$$

On the other hand, the NPV of one unit cost after one cycle can be obtained as

$$\begin{aligned}\delta_b(n_0) &= \sum_{n=0}^{n_0-1}\Big[\sum_{l_1=0}^{(k-1)n} b^{kn} g_1(l_1) + \sum_{l_1=(k-1)n+1}^{\infty} b^{n+l_1} g_1(l_1)\Big]p(n)\\ &\quad + \sum_{n=n_0}^{\infty}\Big[\sum_{l_2=0}^{(k-1)n_0} b^{kn_0} g_2(l_2) + \sum_{l_2=(k-1)n_0+1}^{\infty} b^{n_0+l_2} g_2(l_2)\Big]p(n)\end{aligned}$$

Therefore, the NPV of the expected total cost over the time horizon $[0,\infty)$ is given by

$$TC_b(n_0) = \sum_{n=0}^{\infty} S_b(n_0)\Big\{\delta_b(n_0)\Big\}^n = S_b(n_0)/\,\overline{\delta}_b(n_0). \tag{2}$$

Before we proceed further, we make the following plausible assumptions:

(A-1) $\mu_1^{-1} \geq \mu_2^{-1}$;

(A-2) $c_s d \leq (1-b)\cdot TC_b(n_0)$.

Of our interest is to determine the optimal production time n_0^* which minimizes $TC_b(n_0)$. In order to avoid an unrealistic decision making, we assume that $\underline{n_0} \leq n_0^* \leq \overline{n_0}$ where $\underline{n_0}$ and $\overline{n_0}$ are the lower and upper bounds of n_0, respectively.

Optimality under geometrically distributed repair times :

Suppose that the corrective and preventive repair time distributions are both geometric *i.e.*,

$$g_1(l_1) = \begin{cases} 0, & \text{for } l_1 = 0\\ q_1^{l_1-1}(1-q_1), & \text{for } l_1 = 1,2,3,\cdots;\ 0<q_1<1;\end{cases}$$

$$g_2(l_2) = \begin{cases} 0, & \text{for } l_2 = 0\\ q_2^{l_2-1}(1-q_2), & \text{for } l_2 = 1,2,3,\cdots;\ 0<q_2<1;\end{cases}$$

and $TC_{1b}(n_0) = S_{1b}(n_0)/\,\overline{\delta_{1b}}(n_0)$ denote the corresponding NPV of the expected total cost over an infinite time horizon. Define the numerator of the difference of $TC_{1b}(n_0)$ with respect to n_0, divided by the factor $(1-b)\overline{P}(n_0-1)$ as $W_{1b}(n_0)$. Then we have

$$\begin{aligned}W_{1b}(n_0) &= \Big[\frac{c_1 b^{n_0} r(n_0)}{1-bq_1} + \frac{c_2 b^{n_0}}{1-bq_2}\Big\{bG(n_0)-1\Big\} + \frac{c_i b d}{(1-b)^2}\Big\{kb^{n_0}(1-b) - b^{kn_0}(1-b^k)\Big\}\\ &\quad \times G(n_0) + c_s d b^{kn_0}\Big\{r(n_0)\frac{q_1^{(k-1)n_0}}{1-bq_1} + G(n_0)\frac{bq_2^{(k-1)(n_0+1)}}{1-bq_2} - \frac{q_2^{(k-1)n_0}}{1-bq_2}\Big\}\Big]\frac{\overline{\delta}_b(n_0)}{1-b}\\ &\quad - \Big[r(n_0)\Big\{\frac{q_1^{(k-1)n_0}}{1-bq_1} - \frac{1-b^k}{1-b}\Big\} + G(n_0)\frac{b^k q_2^{(k-1)(n_0+1)}}{1-bq_2} - \frac{q_2^{(k-1)n_0}}{1-bq_2} + \frac{1-b^k}{1-b}\Big]\\ &\quad \times b^{kn_0} S_{1b}(n_0),\end{aligned} \tag{3}$$

where $r(n) = p(n)/\,\overline{P}(n-1)$ is the hazard (failure) rate of the discrete failure distribution and $G(n) = \overline{P}(n)/\,\overline{P}(n-1) = 1 - r(n)$. We characterize the optimal production policy in the following theorem:

Theorem 1. *Suppose that the lifetime distribution $P(n)$ is IFR (increasing failure rate). If (i) $W_{1b}(\underline{n_0}) < 0$ and $W_{1b}(\overline{n_0}) > 0$ then there exists at least one (and at most two) local optimal* *solution* n_0^* $(0 < \underline{n_0} < n_0^* < \overline{n_0} < \infty)$ *satisfying* $W_{1b}(n_0^* - 1) < 0$ *and* $W_{1b}(n_0^*) \geq 0$, *provided that the following conditions including the conditions (A-1) and (A-2) hold:*

(A-3) $\left(\frac{c_1}{1-bq_1} - \frac{c_2}{1-bq_2}\right)\{b\Delta r(\underline{n_0}) - (1-b)r(\overline{n_0})\} + \frac{c_2(1-b)^2}{1-bq_2} \geq 0.$

(A-4) $G(\overline{n_0})\left[b^{(k-1)\overline{n_0}}\left(\frac{1-b^k}{1-b}\right)^2 - k\right] - \frac{b}{(1-b)^2}\left[k(1-b) - (1-b^k)b^{(k-1)(\overline{n_0}+1)}\right]\ \Delta r(\overline{n_0}) \geq 0.$

(A-5) $\phi_b(q_1;\underline{n_0},\overline{n_0}) \geq b^k q_2^{k-1}\ \phi_b(q_2;\overline{n_0},\underline{n_0}) + q_2^{(k-1)\underline{n_0}}(1-b^k q_2^{k-1})^2/(1-bq_2),$

where $\phi_b(q;x,y) = q^{(k-1)y}\left\{r(x)\left(1-b^k q^{k-1}\right) - \Delta r(y) b^k q^{k-1}\right\}/(1-bq).$

(ii) If $W_{1b}(\overline{n_0}) \leq 0$ *then* $n_0^* = \overline{n_0}$. *On the other hand, if* $W_{1b}(\underline{n_0}) \geq 0$ *then* $n_0^* = \underline{n_0}$.

Proof. Since $\Delta r(n) = -\Delta G(n)$ and $0 < q_1, q_2 < 1$, it can be shown that when $P(n)$ is IFR, $\Delta W_{1b}(n_0) > 0$ for all $n_0 \in [\underline{n_0}, \overline{n_0}]$ under assumptions (A-1)-(A-5). Hence the proof of the first part of the theorem is straightforward. For the second part, if $W_{1b}(\overline{n_0}) \leq 0$ then $TC_{1b}(n_0)$ is decreasing in the interval $[\underline{n_0}, \overline{n_0}]$ and therefore, $n_0^* = \overline{n_0}$. On the other hand, if $W_{1b}(\underline{n_0}) \geq 0$ then $TC_{1b}(n_0)$ is increasing in the interval $[\underline{n_0}, \overline{n_0}]$ and therefore, $n_0^* = \underline{n_0}$. Hence, the proof is completed.

3. The Long-run Average Cost Model

Using l'Hospital's rule, the long-run average cost in the steady state $C(n_0)$ can be obtained as

$$\lim_{b \to 1}\left\{(1-b)\cdot TC_b(n_0)\right\} = S(n_0)/T(n_0) = C(n_0), \tag{4}$$

where the mean time length of one cycle is

$$\begin{aligned} T(n_0) &= \sum_{n=0}^{n_0-1}\left[\sum_{l_1=0}^{(k-1)n} kn\ g_1(l_1) + \sum_{l_1=(k-1)n+1}^{\infty}(n+l_1)g_1(l_1)\right] p(n) \\ &+ \sum_{n=n_0}^{\infty}\left[\sum_{l_2=0}^{(k-1)n_0} kn_0\ g_2(l_2) + \sum_{l_2=(k-1)n_0+1}^{\infty}(n_0+l_2)g_2(l_2)\right] p(n) \end{aligned} \tag{5}$$

and the expected cost per cycle is

$$\begin{aligned} V(n_0) &= c_0 + c_1\mu_1^{-1}\sum_{n=0}^{n_0-1} p(n) + c_2\mu_2^{-1}\sum_{n=n_0}^{\infty} g_2(l_2)\ p(n) + \frac{c_i d(k-1)k}{2}\left[\sum_{n=0}^{n_0-1} n^2 p(n)\right. \\ &\left. + \sum_{n=n_0}^{\infty} n_0^2\ p(n)\right] + c_s d\left[\sum_{n=0}^{n_0-1}\sum_{l_1=(k-1)n+1}^{\infty}\left\{l_1 - (k-1)n\right\}g_1(l_1)\ p(n)\right. \\ &\left. + \sum_{n=n_0}^{\infty}\sum_{l_2=(k-1)n_0+1}^{\infty}\left\{l_2 - (k-1)n_0\right\}g_2(l_2)\ p(n)\right]. \end{aligned} \tag{6}$$

Optimality under geometrically distributed repair times:

For the geometric repair time distributions defined in in the previous section, let $T_1(n_0)$, $V_1(n_0)$ denote the mean time length of a cycle and the expected cost per cycle, respectively. Let

$W_1(n_0)$ denote the numerator of the difference of $C_1(n_0) = V_1(n_0)/T_1(n_0)$ divided by the factor $\overline{P}(n_0 - 1)$. Then we have the following theorem:

Theorem 2. *Suppose that the failure time distribution $P(n)$ is IFR, and that the conditions (A-1), (A-2) as $b \to 1$ and the following conditions hold for the hazard rate $r(n)$.*

(A-6) $1 - r(\overline{n_0}) - (\overline{n_0} + 3/2)\ \Delta r(\overline{n_0})\ \geq 0.$

(A-7) $\phi(q_1; \underline{n_0}, \overline{n_0}) \geq q_2^{k-1}\ \phi(q_2; \overline{n_0}, \underline{n_0}) + q_2^{(k-1)\underline{n_0}}(1 - q_2^{k-1})^2/(1 - q_2), \quad k = 2, 3, 4, \cdots.$

where $\phi(q; x, y) = q^{(k-1)y}\left\{r(x)\left(1 - q^{k-1}\right) - \Delta r(y) q^{k-1}\right\}/(1 - q).$

(i) If $W_1(\underline{n_0}) < 0$ and $W_1(\overline{n_0}) > 0$ then there exists at least one (at most two) local optimal solution n_0^* $(0 < \underline{n_0} < n_0^* < \overline{n_0} < \infty)$ satisfying $W_1(n_0^* - 1) < 0$ and $W_1(n_0^*) \geq 0$. The corresponding minimum expected cost $C_1(n_0^*)$ $(= V_1(n_0^*)/T_1(n_0^*))$ satisfies the inequality $\Phi(n_0^* - 1) < C_1(n_0^*) \leq \Phi(n_0^*)$, where

$$\begin{aligned}
\Phi(n) \quad = \quad & \left[r(n) \left\{ \frac{c_1}{1 - q_1} - \frac{c_2}{1 - q_2} + \frac{c_s}{1 - q_1} q_1^{(k-1)n} \right\} + \frac{c_i d(k-1)k}{2}(2n + 1)G(n) \right. \\
& \left. + \frac{c_s d}{1 - q_2}\left\{ G(n) q_2^{(k-1)(n+1)} - q_2^{(k-1)n} \right\} \right] \Big/ \left[r(n) \left\{ kn + \frac{q_1^{(k-1)n}}{1 - q_1} \right\} \right. \\
& \left. + G(n) \left\{ k(n+1) + \frac{q_2^{(k-1)(n+1)}}{1 - q_2} \right\} - kn - \frac{q_2^{(k-1)n}}{1 - q_2} \right]. \qquad (7)
\end{aligned}$$

(ii) If $W_1(\overline{n_0}) \leq 0$ then $n_0^* = \overline{n_0}$. On the other hand, if $W_1(\underline{n_0}) \geq 0$ then $n_0^* = \underline{n_0}$.

Proof. The proof is omitted for brevity.

Remark. Under geometric failure, assumption (A-6) is trivially true and assumption (A-7) is clearly validated when $q_1, q_2 \to 1$.

4. The NPV Model with Flexible Production Rate

Suppose that $p = kd$, where k, an integer > 1, is a decision variable . Then the associated nonlinear integer programing (NIP) problem can be formulated as follows:

NIP :

min $\quad Z(\mathbf{x})$

s.t. $\quad h_i(\mathbf{x}) \geq b_i, \quad i = 1, 2, 3;$

$\mathbf{x} = [n_0, k], \quad n_0, k$: integer;

where $\quad Z(\mathbf{x}) \equiv TC_b(n_0, k) = S_b(n_0, k)/\ \overline{\delta_b}(n_0, k)$

and $\quad h_1(\mathbf{x}) \equiv -n_0, \quad h_2(\mathbf{x}) \equiv n_0, \quad h_3(\mathbf{x}) \equiv k, \quad b_1 \equiv -\overline{n_0}, \quad b_2 \equiv \underline{n_0}, \quad b_3 \equiv 2.$

For solving the above NIP we develop a hybridized neural network (NN) and branch and bound (BB) algorithm. First we consider that the NIP problem has no integer restrictions, because the neural network technique is an approximate method suitable for continuous values, i.e., we solve the nonlinear programing (NP) problem. We construct the *energy function* based on the penalty function method for solving the NP problem. The penalty function method transforms the constrained optimization problem into the unconstrained optimization one. In order to solve the NP problem, we construct the following energy function:

$$E(\mathbf{x},m) = -Z(\mathbf{x}) + \frac{m}{2}\left[\sum_{i=1}^{3}([b_i - h_i(\mathbf{x})]_-)^2 + ([n_0]_-)^2 + ([k]_-)^2\right], \tag{8}$$

where $m > 0$ is penalty parameter and $[y]_- = \min\{0, y\}$. Minimization of $E(\mathbf{x},m)$ leads to the following system of ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = -\mu\ \nabla_x\ E(\mathbf{x},m),$$

where $\mu\ (\geq 0)$ is called learning parameter. In the following algorithm, we state the overall procedure for solving the NIP.

Hybridized NN and BB algorithm :

Step 1. Construct the energy function $E(\mathbf{x},m)$ based on penalty function method.

Step 2. Using gradient method obtain the system of ordinary differential equations.

Step 3. Set the parameter m, μ and the initial values of n_0 and k.

Step 4. Using **NDSolve function** in Mathematica solve the system of ordinary differential equations. Using **Plot function** in Mathematica draw the graph of the dynamic convergence process.

Step 5. From the NP solution, examine whether n_0^* and k^* are both integers or not. If they are both integers then they give the best solution. Stop. Otherwise, go to Step 6.

Step 6. Choose one variable, say n_0, whose value is not integer and make two separate problems with known integer value of n_0 as $n_0 \leq \lfloor n_0^* \rfloor$ and $n_0 \geq \lceil n_0^* \rceil$.

Step 7. Solve these problems using the NN method and choose the solution which provides the minimum cost.

Step 8. Repeat Step 5 and if needed, Step 6 to find the best feasible solution of the NIP.

5. Numerical Examples

We consider the negative binomial failure distribution with shape parameter 2 and geometric repair time (corrective and preventive) distributions with parameters q_1 and q_2. The p.m.f. of the negative binomial failure time distribution is given by $p(n) = np_0^2q_0^{n-1}$, $n = 0,1,2,3,......;$ $0 < p_0 < 1, q_0 = 1 - p_0$. First we find the optimal results when p is predetermined *i.e.* k is a known integer > 1. We take the parameter values as $d = 90$, $k = 2$, $c_i = 0.5, c_s = 1.25$, $c_0 = 1500$, $c_1 = 200$, $c_2 = 100$, $q_1 = 0.4, q_2 = 0.2$, $b = 0.9$, $\underline{n_0} = 3$, $\overline{n_0} = 8$.

Tables 1 presents the dependence of the optimal production policy on the parameter p_0 in the NPV and average cost models. Note that as p_0 increases, the MTTF (mean time to failure) decreases and hence the expected cost rate increases gradually. Table 2 shows that the NPV of the expected total cost is almost insensitive to changes in value of the preventive repair cost parameter c_2 while a low impact on the expected cost rate is observed for changes in the value of the corrective repair cost parameter c_1. The asymptotic behavior of the NPV function $TC_{1b}(n_0)$ is reflected in Table 3. When k is treated as a decision variable, the optimal results obtained by the hybridized NN and BB algorithm are found to be superior to those of the model with inflexible production rate, see Table 4.

Table 1 Influence of p_0 on the optimal production policy

	NPV model		Average cost model	
p_0	n_0^*	$TC_{1b}(n_0^*)$	n_0^*	$C_1(n_0^*)$
0.1	6	3483.35	6	275.861
0.2	6	3650.85	6	290.452
0.3	6	3907.44	6	313.703
0.4	6	4249.29	7	346.122
0.5	6	4674.53	7	388.827
0.6	6	5179.00	7	442.360
0.7	5	5754.32	7	505.516
0.8	4	6391.05	6	576.283
0.9	3	7076.57	3	652.461

Table 2 Dependence of the optimal production policy on the parameters c_1 and c_2 in the NPV model when $p_0 = 0.5$.

	$(c_1, c_2) = (R, 100)$		$(c_1, c_2) = (200, R)$	
R	n_0^*	$TC_{1b}(n_0^*)$	n_0^*	$TC_{1b}(n_0^*)$
100	7	4426.12	6	4674.53
120	7	4476.68	6	4677.75
140	6	4526.98	6	4680.98
160	6	4576.17	7	4684.20
180	6	4625.35	7	4685.52
200	6	4674.53	7	4687.17

Table 3 Asymptotic behavior of $TC_{1b}(n_0)$ when $p_0 = 0.5$

b	n_0^*	$(1-b)TC_{1b}(n_0^*)$
0.900000	6	467.453
0.990000	7	396.310
0.999000	7	389.571
0.999900	7	388.901
0.999990	7	388.834
0.999999	7	388.826

Table 4 Optimal results of the NPV model when p is predetermined or flexible

	$p\ (= kd)$ is flexible			$p\ (= 2d)$ is predetermined	
p_0	k^*	n_0^*	$TC_b(k^*, n_0^*)$	n_0^*	$TC_{1b}(n_0^*)$
0.1	2	6	3483.35	6	3483.35
0.2	2	6	3650.85	6	3650.85
0.3	2	6	3907.44	6	3907.44
0.4	2	6	4249.29	6	4249.29
0.5	3	3	4476.80	6	4674.53
0.6	4	3	4668.88	6	5179.00
0.7	4	3	4843.38	5	5754.32
0.8	5	3	4944.55	4	6391.05
0.9	5	3	5049.92	3	7076.57

We now make an attempt to derive approximately the production policy for discrete time setting from the continuous time model. For this, note that the negative binomial failure distribution defined at the beginning of this section corresponds to the continuous time gamma distribution (shape parameter 2) $F(t) = 1 - (1 + \mu_0 t)\exp(-\mu_0 t), \quad \mu_0 > 0$. Further, the mean value $2/p_0$ for the negative binomial distribution corresponds to the mean value $2/\mu_0$ for the gamma distribution. Utilizing this correspondence and $\exp(-\beta t) = b^t$, *i.e.*, $\beta = ln(1/b)$, we consider the continuous time model with gamma failure distribution (shape parameter 2) and uniform repair time distributions:

$$U_1(s_1) = \begin{cases} \frac{s_1}{n_1}, & 0 \le s_1 \le n_1 \\ 1, & s_1 > n_1, \end{cases}$$

$$U_2(s_2) = \begin{cases} \frac{s_2}{n_2}, & 0 \le s_2 \le n_2 \\ 1, & s_2 > n_2, \end{cases}$$

and derive the approximate policy by time discretization. The data are taken as $d = 90,\ k = 2,\ c_i = 0.5, c_s = 1.25,\ c_0 = 1500,\ c_1 = 200,\ c_2 = 100, n_1 = 12, n_2 = 8, b = 0.9, \underline{n_0} = 3,\ \overline{n_0} = 8$. Table 5 exhibits that time discretization from the outcome of continuous time model results a remarkable difference in terms of the failure rate and lot sizing decision.

Table 5 A comparison of the results of discrete optimization and those of discrete time approximation from the continuous time NPV model

p_0	Continuous time model		Discrete time approximation		Discrete optimization	
(μ_0)	t_0^*	$TC_b(t_0^*)$	η_0^*	$TC_b(\eta_0^*)$	n_0^*	$TC_b(n_0^*)$
0.1	5.033	3679.32	5	3679.37	6	3600.54
0.2	4.253	3904.15	4	3908.34	5	3752.54
0.3	3.720	4114.99	4	4121.00	5	3944.18
0.4	3.336	4299.57	3	4310.79	5	4162.63
0.5	3.044	4459.04	3	4459.25	4	4387.20
0.6	2.814	4596.96	3	4600.55	4	4607.69
0.7	2.628	4716.91	3	4731.01	3	4818.91
0.8	2.474	4821.95	3	4849.03	$3^\dagger$	5020.73
0.9	2.343	4914.59	$3^\dagger$	4954.36	$3^\dagger$	5215.24

† indicates $n_0^* = \underline{n_0}$

6. Concluding Remarks

In this paper, we have modeled a general EMQ problem with stochastic machine breakdown and repair in a discrete-time framework. The NPV of the expected total cost function is derived under general discrete failure and discrete repair time distributions. Both the discounted cost criteria and long-run average cost criteria for the existence of a local optimal solution are derived under geometric repair time distributions. From the numerical study we have observed that (i) the NPV approach is superior to the long-run average cost approach and (ii) time discretization from the continuous time model results a remarkable difference in the lot sizing policy in comparison to that of the actual discrete time optimization. In developing the model, we have assumed that a failure, if occurs during a production phase, can be detected immediately and perfectly. However, this may be unrealistic in some real manufacturing systems. Moreover, for discrete time setting, we have taken the production-demand ratio as an integer greater than 1. Future research could be carried out to relax these assumptions. Another direction may be to extend the model for the system with multiple machines.

References

Abboud, N.E. (2001) A discrete-time Markov production-inventory model with machine breakdowns. *Computers and Industrial Engineering*, **39**, 95-107.

Bielecki, T. and Kumar, P.R. (1988) Optimality of zero-inventory policies for unreliable manufacturing systems. *Operations Research*, **36**, 532-541.

Cheung, K.L. and Hausman, W.H. (1997) Joint determination of preventive maintenance and safety stocks in an unreliable production environment. *Naval Research Logistics*, **44**, 257-271.

Dohi, T., Yamada, Y., Kaio, N. and Osaki, S. (1997) The optimal lot sizing for unreliable economic manufacturing model. *International Journal of Reliability, Quality and Safety Engineering*, **4**, 413-426.

Dohi, T., Okamura, H. and Osaki, S. (2001) Optimal control of preventive maintenance schedule and safety stocks in an unreliable manufacturing environment. *International Journal of Production Economics*, **74**, 147-155.

Groenevelt, H., Pintelon, L. and Seidmann, A. (1992a) Production lot sizing with machine breakdowns. *Management Science*, **38**, 104-123.

Groenevelt, H., Pintelon, L. and Seidmann, A. (1992b) Production batching with machine breakdowns and safety stocks. *Operations Research*, **40**, 959-971.

Kim, C.H. and Hong, Y. (1997) An extended EMQ model for a failure prone machine with general lifetime distribution. *International Journal of Production Economics*, **49**, 215-223.

Liu, B. and Cao, J. (1999) Analysis of a production-inventory system with machine breakdowns and shutdowns. *Computers and Operations Research*, **26**,73-91.

Makis, V. and Fung, J. (1998) An EMQ model with inspections and random machine failures. *Journal of the Operational Research Society*, **49**, 66-75.

McCall, J.J. (1965) Maintenance policies for stochastically failing equipment: a survey. *Management Science*, **11**, 493-524.